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Time Dependence of Joint Entropy of Oscillating Quantum Systems

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Abstract The time dependent entropy (or Leipnik's entropy) of harmonic and damped harmonic oscillator systems is studied by using time dependent wave function obtained by the Feynman path integral method. The Leipnik entropy and its envelope change as a function of time, angular frequency and damping factor. Our results for simple harmonic oscillator are in agreement with the literature. However, the joint entropy of damped harmonic oscillator shows remarkable discontinuity with time for certain values of damping factor. The envelope of the joint entropy curve increases with time monotonically. These results show the general properties of the envelope of the joint entropy curve for quantum systems.

Keywords Joint entropy · Simple harmonic oscillator · Damped harmonic oscillator · Path integral

1 Introduction

The investigation of time dependent entropy of the quantum mechanical systems attracts much attention in recent years. For both open and closed quantum systems, the different information-theoretic entropy measures have been discussed [1–8]. In addition to these measures, the joint entropy [9, 10] can also be used to measure the loss of information related to evolving pure quantum states [11]. The joint entropy of the physical systems named as MACS (maximal classical states) were conjectured by Dunkel and Trigger [12]. They have

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showed that the joint entropy of the quantum mechanical systems increases monotonically with time but these results are not sufficient for simple harmonic oscillator [13].

The aim of this study is to calculate the complete joint entropy information analytically for simple and damped quantum harmonic oscillators. Unlike in classical (Shannon) information theory, the quantum (von Neumann) conditional entropy can be negative when quantum entangled systems considered which is named as quantum non-separability. Thus the possibility that the negative (virtual) information can be carried by entangled particles indicates a consistent interpretation of quantum informational processes [14]. On the other hand, the Wigner function [15, 16] can be used to calculate quantum corrections for classical thermodynamical quantities (in the high temperature limit). However, in quantum mechanics the Wigner function takes negative values in general. This is settled by using the Wehrl entropy. So it can not provide a satisfactory result for our purpose to calculate the loss of information related with the quantum evolution of pure states. For this reason, the Wehrl entropy can not represent a "fundamental" entropy definition for quantum systems. Hence, not only the von Neumann entropy but also the Wehrl entropy does not provide a sufficient information-theoretic measures for this purpose. Therefore, we used in this study a different entropy measure named as Leipnik's or joint entropy. We have calculated the Leipnik entropy for simple and damped quantum harmonic oscillators. It is shown how it (and its envelope) changes as a function of time, angular frequency and damping factor. This is a relatively hot subject, which makes use of quantum entropy arguments in order to shed light on the behavior of the physical system under investigation. In the context of this paper, what is even more valuable is to assess whether the Leipnik entropy can be used as a measure of the loss of information associated with the quantum evolution of the states. The original proposal introduced by Dunkel and Trigger is that Leipnik entropy can be considered as a general measure so that it increases in time (corresponding to the information loss in the system) for "all" quantum systems.

This paper is organized as follows. In Sect. 2, we explained fundamental definitions needed for the calculations. In Sect. 3, we presented calculations and results. Moreover, we obtained the analytical solution of Kernel, wave function in both coordinate and momentum spaces and its joint entropy. We also obtained same quantities for damped harmonic oscillator case. Finally, we presented the conclusion in Sect. 4.

2 Fundamental Definitions

We deal with a classical system with $d = sN$ degrees of freedom, where N is the particle number and s is number of spatial dimensions [12]. We assume that the density function of a system $g(x, p, t) = g(x_1, \dots, x_d, p_1, \dots, p_d, t)$ in phase space is non-negative, time dependent and normalized to unity,

$$\int dx\, dpg(x, p, t) = 1. \tag{1}$$

The Gibbs–Shannon entropy is described by

$$S(t) = -\frac{1}{N!} \int dx\, dpg(x, p, t) \ln(h^d g(x, p, t)), \tag{2}$$

where $h = 2\pi\hbar$ is the Planck constant. Schrödinger wave equation with the Born interpretation [17] is given by

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi. \tag{3}$$

The quantum probability densities are defined in position and momentum spaces as $|\psi(x,t)|^2$ and $|\tilde{\psi}(p,t)|^2$, where $\tilde{\psi}(p,t)$ is given as,

$$\tilde{\psi}(p,t) = \int \frac{dx\, e^{-ipx/\hbar}}{(2\pi\hbar)^{d/2}} \psi(x,t). \tag{4}$$

Leipnik proposed the product function as [12]

$$g_j(x,p,t) = |\psi(x,t)|^2 |\tilde{\psi}(p,t)|^2 \geq 0. \tag{5}$$

Substituting (5) into (2), we get the joint entropy $S_j(t)$ for the pure state $\psi(x,t)$ or equivalently it can be written in the following form [12]

$$S_j(t) = -\int dx |\psi(x,t)|^2 \ln |\psi(x,t)|^2 - \int dp |\tilde{\psi}(p,t)|^2 \ln |\tilde{\psi}(p,t)|^2 - \ln h^d. \tag{6}$$

We find time dependent wave function by means of the Feynman path integral [18]

$$\begin{aligned} K(x'',t'';x',t') &= \int_{x'=x(t')}^{x''=x(t'')} Dx(t) e^{\frac{i}{\hbar} S[x(t)]} \\ &= \int_{x'}^{x''} Dx(t) e^{\frac{i}{\hbar} \int_{t'}^{t''} L[x,\dot{x},t] dt}. \end{aligned} \tag{7}$$

The Feynman kernel can be related to the time dependent Schrödinger's wave function

$$K(x'',t'';x',t') = \sum_{n=0}^{\infty} \psi_n^*(x',t') \psi_n(x'',t''). \tag{8}$$

The propagator in semiclassical approximation reads

$$K(x'',t'';x',t') = \left[\frac{i}{2\pi\hbar} \frac{\partial^2}{\partial x' \partial x''} S_{cl}(x'',t'';x',t') \right]^{1/2} e^{\frac{i}{\hbar} S_{cl}(x'',t'';x',t')}. \tag{9}$$

The prefactor is often referred to as the Van Vleck-Pauli-Morette determinant [19, 20]. The $F(x'',t'';x',t')$ is given by

$$F(x'',t'';x',t') = \left[\frac{i}{2\pi\hbar} \frac{\partial^2}{\partial x' \partial x''} S_{cl}(x'',t'';x',t') \right]^{1/2}. \tag{10}$$

3 Calculation and Results

3.1 Simple Harmonic Oscillator (SHO)

To get the path integral solution for the SHO, we must calculate its action function. The Lagrangian of the system is given by

$$L(x,\dot{x},t) = \frac{m}{2}\left(\dot{x}^2 - \frac{1}{2}\omega^2 x^2 \right). \tag{11}$$

Following a straightforward calculation, we get

$$S(x_{cl}(t''), x_{cl}(t')) = \frac{m\omega}{2\sin\omega t}[(x_{cl}''^2 + x_{cl}'^2)\cos\omega t - 2x_{cl}'x_{cl}''] \tag{12}$$

with $t = t'' - t'$ and $x_{cl}' = x_0, x_{cl}'' = x$. Substituting (12) into (9), we obtain the Feynman kernel [18]:

$$K(x, x_0; t) = \left(\frac{m\omega}{2\pi\hbar i \sin\omega t}\right)^{\frac{1}{2}} \exp\left\{-\frac{m\omega}{2i\hbar}\left[(x^2 + {x_0}^2)\cot\omega t - \frac{2x_0 x}{\sin\omega t}\right]\right\}. \tag{13}$$

By using the Mehler-formula, we have

$$e^{-(x^2+y^2)/2}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{z}{2}\right)^2 H_n(x)H_n(y) = \frac{1}{\sqrt{1-z^2}}\exp\left[\frac{4xyz - (x^2+y^2)(1+z^2)}{2(1-z^2)}\right] \tag{14}$$

where H_n is Hermite polynomials, we can write the Feynman kernel defining $x \equiv \sqrt{m\omega/\hbar}x_0$, $y \equiv \sqrt{m\omega/\hbar}x$ and $z = e^{-i\omega T}$ as

$$K(x, x_0; t) = \sum_{n=0}^{\infty} e^{-itE_n/\hbar}\Psi^*(x_0)\Psi(x). \tag{15}$$

Thus we obtain the energy eigenvalues

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right), \tag{16}$$

and the corresponding wave functions

$$\Psi_n(x) = \left(\frac{m\omega}{2^{2n}\pi\hbar n!^2}\right)^{\frac{1}{4}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)\exp\left(-\frac{m\omega}{2\hbar}x^2\right). \tag{17}$$

Time-dependent wave function of the SHO is defined as

$$\Psi(x, t) = \int K(x, x_0; t)\Psi(x_0, 0)dx_0. \tag{18}$$

It can be written as

$$\Psi(x, t) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left\{-\frac{\bar{\alpha}}{4} - \frac{\alpha^2}{2} - \frac{i\omega t}{2}\right\}\exp\left[-\frac{\bar{\alpha}^2}{4}e^{-2i\omega t} + \alpha\bar{\alpha}e^{-i\omega t}\right] \tag{19}$$

where $\bar{x}$ or $\bar{\alpha}$ is mean of the Gaussian curve. The probability density has

$$|\Psi(x, t)|^2 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\exp\left[-(\alpha - \bar{\alpha}\cos\omega t)^2\right] \tag{20}$$

where $\alpha = \sqrt{\frac{m\omega}{\hbar}}x$. Thus it can be rewritten as

$$|\Psi(x, t)|^2 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\exp\left[-\frac{m\omega}{\hbar}(x - \bar{x}\cos\omega t)^2\right]. \tag{21}$$

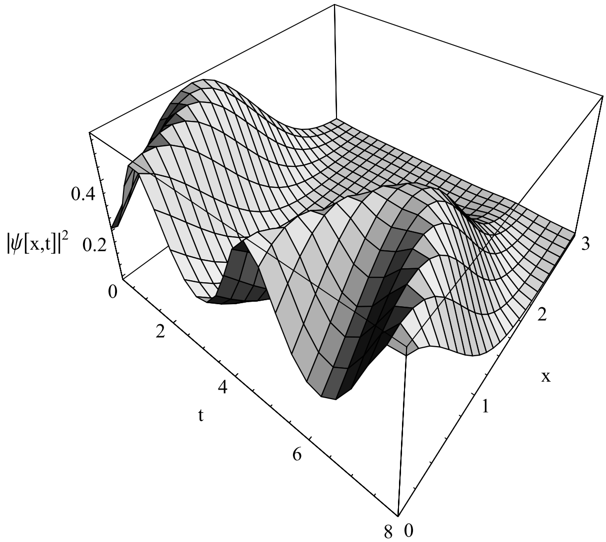


Fig. 1 $|\Psi(x,t)|^2$ versus time and coordinate

This has been shown in Fig. 1. In momentum space, the probability density has the form

$$|\tilde{\psi}(p,t)|^2 = \left(\frac{1}{m\omega\pi\hbar}\right)^{1/2} \exp\left[\frac{-p^2}{m\omega\hbar} + \frac{m\omega\bar{x}^2}{2\hbar}\big(\cos 2\omega(t) - 1\big) - \frac{2p\bar{x}}{\hbar}\sin\omega(t)\right]. \tag{22}$$

The joint entropy of harmonic oscillator becomes

$$S_j(t) = \ln\frac{e}{2} + \frac{4m\omega}{\hbar}\bar{x}^2\sin^2\omega(t). \tag{23}$$

In Fig. 2, the joint entropy of this system is plotted by using Mathematica in three dimensional space. It is seen that displacement of simple harmonic oscillator from equilibrium depends on harmonic functions (e.g. sine or cosine function). Therefore, other properties of the SHO systems indicate the same harmonic behavior. If the frequency of the SHO is sufficiently small, the system shows the same behavior as the free particle [12]. As seen from Figs. 3 and 4, the envelope of the sinusoidal curve monotonically increases with ω and becomes constant when the angular frequency is constant. As the frequency increases, the joint entropy of this system indicates a fluctuation with increasing amplitude with time. If the time goes to zero, it is important that (23) is in agreement with following general inequality for the joint entropy:

$$S_j(t) \geq \ln\left(\frac{e}{2}\right) \tag{24}$$

originally derived by Leipnik for arbitrary one-dimensional one-particle wave functions.

3.2 Damped Harmonic Oscillator (DHO)

The quantum damped oscillator is an important model for many physical systems to describe the interaction of the system with its environment. The Lagrangian of the DHO is given by

$$L(x,\dot{x},t) = e^{\gamma t}\left(\frac{m}{2}\dot{x}^2 - \frac{m}{2}\omega^2 x^2 + j(t)x\right). \tag{25}$$

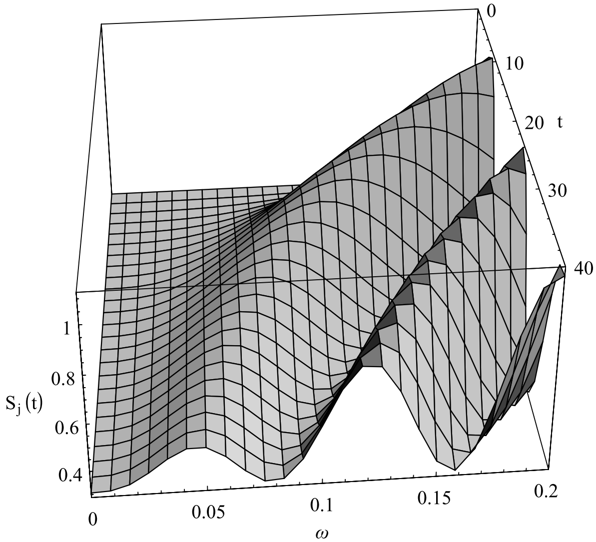


Fig. 2 The 3D graph of joint entropy of simple harmonic oscillator

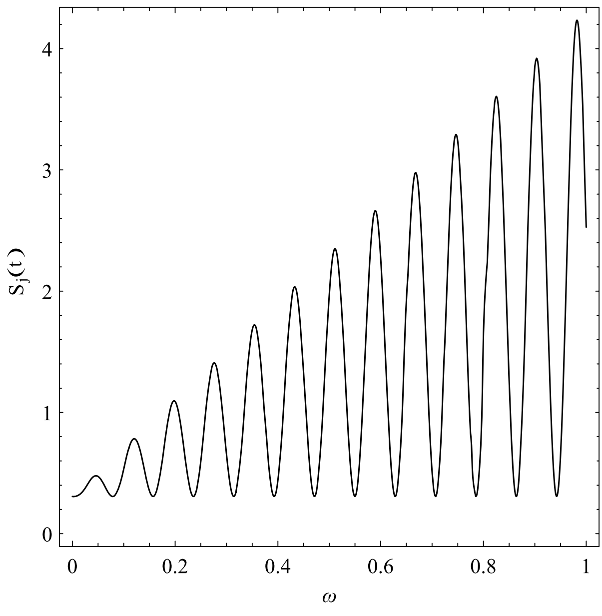


Fig. 3 The joint entropy of simple harmonic oscillator versus ω

Damped free particle kernel is

$$K(x,t;x_0,0)=\left(\frac{\gamma m e^{\gamma t/2}}{4\pi i\hbar\sinh\frac{1}{2}\gamma t}\right)^2\exp\left(\frac{i\gamma m e^{\gamma t/2}}{4\hbar\sinh\frac{1}{2}\gamma t}(x-x_0)^2\right). \tag{26}$$

The DHO kernel has the form [21]

$$K(x,t;x_0,0)=\left(\frac{m\omega e^{\gamma t/2}}{2\pi i\hbar\sinh\omega t}\right)^{1/2}\exp\left(\frac{i}{\hbar}S_{cl}(x,x_0,t)\right), \tag{27}$$

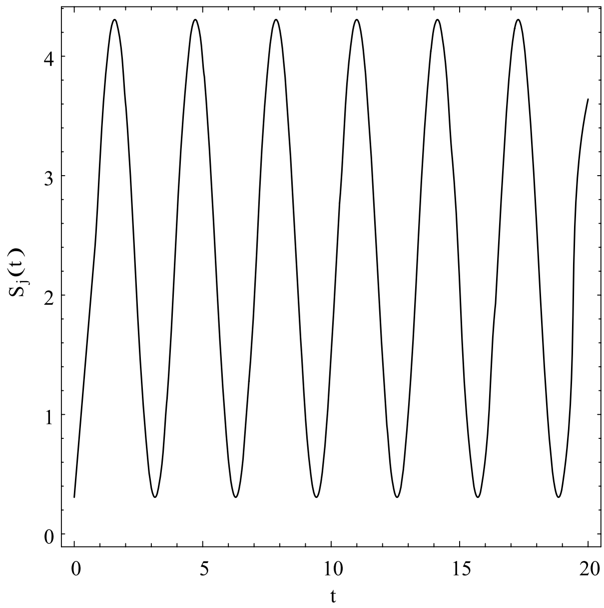


Fig. 4 The joint entropy of simple harmonic oscillator versus time

or explicitly

$$K(x,t;x_0,0) = \left(\frac{m\omega e^{\gamma t/2}}{2\pi i\hbar \sin\omega t}\right)^{1/2} \exp\left[\frac{im}{2\hbar}(ax^2+2bx_0^2+2xx_0c+2xd+2x_0e-f)\right]. \quad (28)$$

Where the coefficients a, b, c, d, e, f are [21]

$$a=\left(-\frac{\gamma}{2}+\omega\cot\omega t\right)e^{\gamma t}, \quad (29)$$

$$b=\left(\frac{\gamma}{2}+\omega\cot\omega t\right), \quad (30)$$

$$c=\left(-\frac{\omega}{\sin\omega t}e^{\gamma t}\right), \quad (31)$$

$$d=\frac{e^{\gamma t}}{m\sin\omega t}\int_0^t j(t')e^{\gamma t'/2}\sin\omega t'\,dt', \quad (32)$$

$$e=\frac{1}{m\sin\omega t}\int_0^t j(t')e^{\gamma t'/2}\sin\omega(t-t')dt', \quad (33)$$

$$f=\frac{1}{m^2\omega}\int_0^t\int_0^{t'} j(t')j(s)e^{\gamma(s+t'/2)}\sin\omega(t-t')\sin\omega s\,ds\,dt'. \quad (34)$$

The wave function $\psi_n(x,0)$ and energy eigenvalues become

$$\psi_n(x,0)=N_0H_n(\alpha_0 x)\exp\left[-\frac{1}{2}\alpha_0 x^2\right] \quad (35)$$

and

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega_0 \tag{36}$$

where $H_n(x)$ is the Hermite polynomial of order n and the coefficients are

$$\alpha_0 = \left(\frac{m\omega}{\hbar}\right)^{1/2}, \qquad N_0 = \frac{\alpha^{1/2}}{(2^n n!\sqrt{\pi})^{1/2}}. \tag{37}$$

The time-dependent wave function is obtained as [21]

$$\begin{aligned}\psi_n(x,t) &= \int_{-\infty}^{\infty} dx_0 K(x,t;x_0,0)\psi(x,0)\\ &= N\frac{1}{(2^n n!)^{1/2}}\exp\left\{-i\left[\left(n+\frac{1}{2}\right)\cot^{-1}\left(\frac{\gamma}{2\omega}+\cot\omega t + f\right)\right]\right\}\\ &\quad \times \exp[-(Ax^2 + +2Bx)]H_n[D(x-E)].\end{aligned} \tag{38}$$

To simplify the evaluation, we set $j(t) = 0$. Therefore kernel and wave function of the DHO [22] become

$$\begin{aligned}K(x,t;x_0,0) = \left(\frac{m\omega e^{\gamma t/2}}{2\pi i\hbar \sin\omega t}\right)^{1/2} \exp\bigg[\frac{im}{4\hbar}\bigg(\gamma(x_0^2 - e^{\gamma t}x^2)\\ + \frac{2\omega}{\sin\omega t}[(x_0^2 + x^2 e^{\gamma t})\cos\omega t - 2e^{\gamma t/2}xx_0]\bigg)\bigg]\end{aligned} \tag{39}$$

where $\omega = (\omega_0^2 - \gamma^2/4)^{1/2}$ and

$$\psi_n(x,t) = \frac{N}{(2^n n!)^{1/2}}\exp\left\{-i\left[\left(n+\frac{1}{2}\right)\cot^{-1}\left(\frac{\gamma}{2\omega}+\cot\omega t\right)\right]\right\}H_n[Dx]\exp[-Ax^2]. \tag{40}$$

Where D, A and N are

$$D(t) = \frac{\alpha e^{\gamma t/2}}{\eta(t)\sin\omega t}, \tag{41}$$

$$\eta^2(t) = \frac{\gamma^2}{4\omega^2} + \frac{\gamma}{\omega}\cos\omega t + \csc^2\omega t, \tag{42}$$

$$A(t) = \frac{m\omega}{2\hbar}e^{\gamma t}\left[\frac{1}{\eta^2(t)\sin^2\omega t} + i\left(\frac{\gamma}{2\omega} - \cot\omega t + \frac{\gamma/2\omega + \cot\omega t}{\eta^2\sin^2\omega t}\right)\right], \tag{43}$$

and

$$N(t) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{\exp(\frac{\gamma t}{4})}{\eta(t)(\sin\omega t)^{1/2}}. \tag{44}$$

The ground state wave function is given by

$$\psi_0(x,t) = N(t)\exp\left\{-i\left[\left(\frac{1}{2}\right)\cot^{-1}\left(\frac{\gamma}{2\omega}+\cot\omega t\right)\right]\right\}\exp[-A(t)x^2]. \tag{45}$$

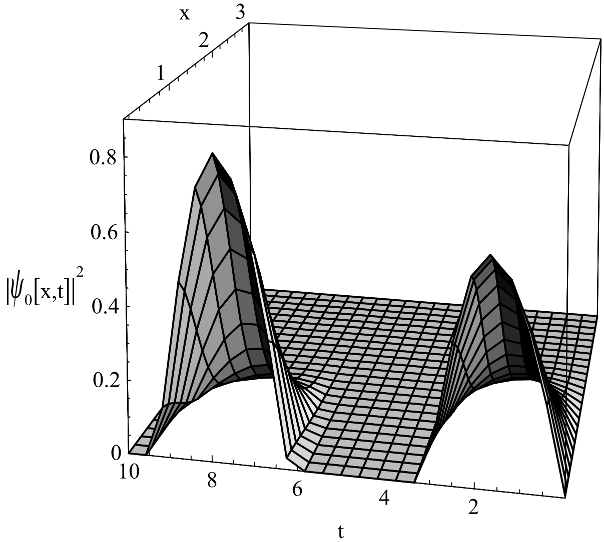


Fig. 5 The probability function as a function of time and coordinate at $\gamma = 0.1$

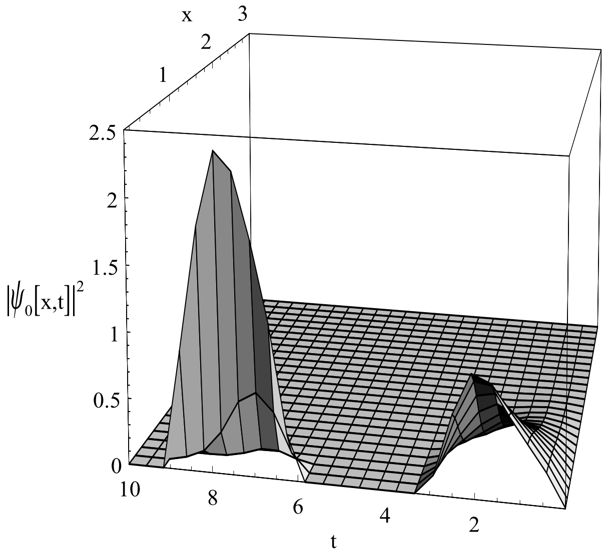


Fig. 6 The probability function as a function of time and coordinate at $\gamma = 0.5$

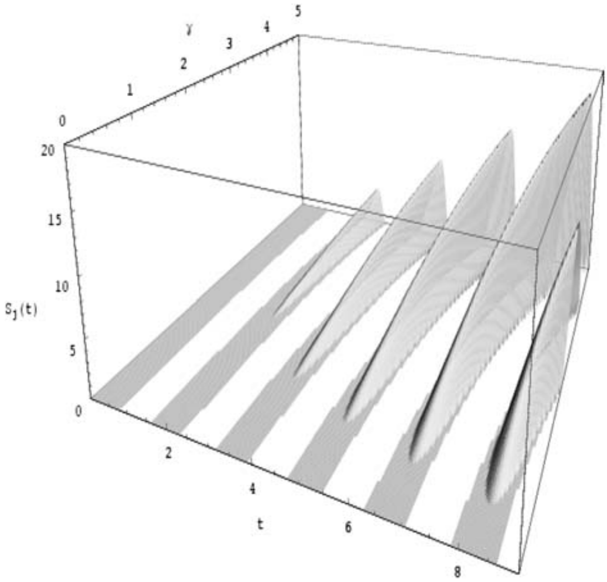
So the probability distribution in coordinate space becomes

$$|\psi_0(x,t)|^2 = N(t)^2 \exp[-2A'(t)x^2] \tag{46}$$

where A' is defined by

$$A'(t) = \frac{m\omega}{2\hbar} e^{\gamma t} \left[\frac{1}{\eta^2(t) \sin^2 \omega t} \right]. \tag{47}$$

Fig. 7 The 3D graph of the joint entropy of damped harmonic oscillator for damping factor (γ) at $\omega_0 = 2$



The probability density in coordinate space is shown in Figs. 5 and 6 for different values of γ. The probability density in momentum space can be written easily

$$|\psi_0(p,t)|^2 = \frac{N(t)^2}{\sqrt{2A(t)A(t)^\dagger \hbar}} \exp\left[-\frac{p^2}{2\hbar^2}\frac{A'(t)}{A(t)A(t)^\dagger}\right]. \tag{48}$$

The time-dependent joint entropy can be obtained from (2) as

$$S_j(t) = N(t)^2\sqrt{\frac{\pi}{2A'(t)}}\left[\left(\ln N(t)^2 - \frac{1}{2}\right) - \frac{1}{2}\sqrt{\frac{1}{2A(t)A(t)^\dagger}}\left(\ln\frac{N(t)^2}{2A(t)A(t)^\dagger} - \frac{1}{2}\right)\right] - \ln 2\pi. \tag{49}$$

The joint entropy depends on damping factor γ. When $\gamma \to 0$, all of the above results are reduced to simple harmonic oscillator case. However, when the $\gamma \neq 0$, the joint entropy has remarkably different features from the SHO. As can be seen in Figs. 7 and 8, the joint entropy of the DHO has very interesting properties. For example, the joint entropy attains negative values for small γ although it is expected to be positive and monotonically increasing in all cases. In addition, the joint entropy associated with this system exhibits periodic discontinuities. On the other hand, the envelope of the joint entropy curve monotonically increases for large values of γ (in Fig. 9). Therefore, the envelope of the joint entropy curves is observed to be monotonically increasing rather than the joint curve itself.

4 Conclusion

We have investigated the joint entropy for explicit time dependent solution of one-dimensional harmonic oscillators. The time dependent wave function is obtained by means

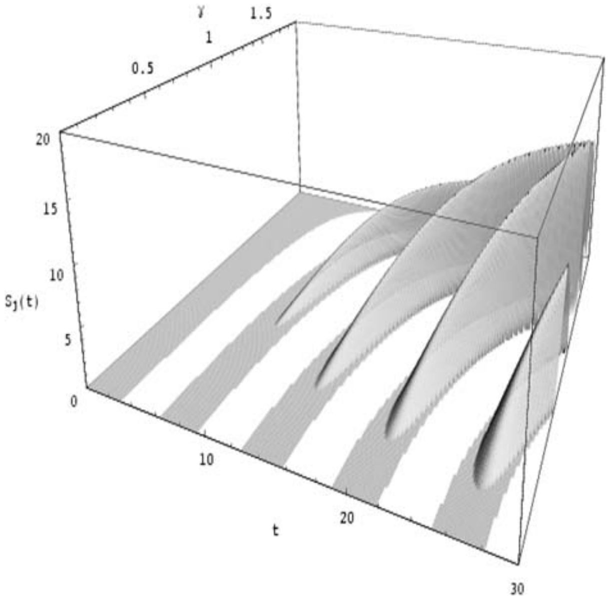


Fig. 8 The 3D graph of the joint entropy of damped harmonic oscillator for damping factor (γ) at $\omega_0 = 1$

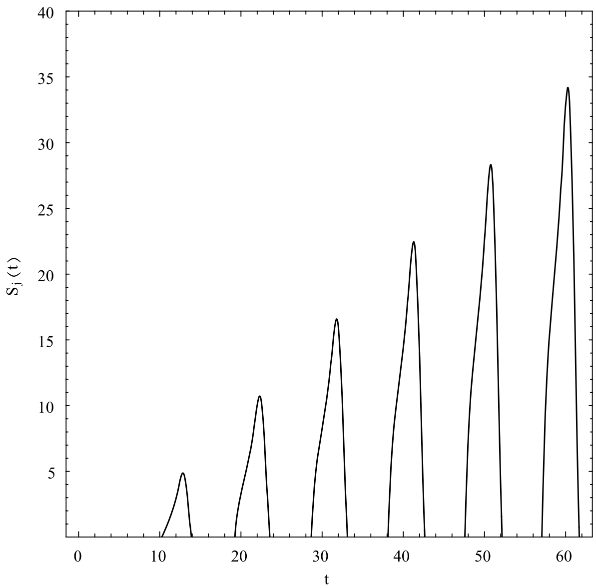


Fig. 9 The 2D graph of the joint entropy of damped harmonic oscillator as a function of time at large (γ)

of Feynmann Path integral technique. Our results show that in the simple harmonic oscillator case, the joint entropy fluctuates with time and frequency. This result indicates that the information is periodically transferred.

On the other hand, the joint entropy curve of the DHO shows remarkable discontinuities with time. This contradicts with the proposal set forth by Dunkel and Trigger. This point has been noticed by Garbaczewski for the first time [13]. However, Garbaczewski reached this conclusion for the cases such that the associated Leipnik entropy values do not increase with time or changes periodically in time. It also depends on choice of initial values of parameter i.e. ω. These results can be explained as the information exchange between harmonic oscillator and the system providing damping. But the information exchange appears in certain values of time for damping. If the damping factor increases, the information entropy has not periodicity anymore. Moreover, for certain values of the damping factor, the transfer of information between systems is exhausted.

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